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A Note on Equilibrium Problems with Properly Quasimonotone Bifunctions

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Abstract. In this paper, we consider some well–known equilibrium problems and their duals in a topological Hausdorff vector space X for a bifunction F defined on $K \times K$,where K is a convex subset of X. Some necessary conditions are investigated, proving different results depending on the behaviour of F on the diagonal set. The concept of proper quasimonotonicity for bifunctions is defined, and the relationship with generalized monotonicity is investigated. The main result proves that the condition of proper quasimonotonicity is sharp in order to solve the dual equilibrium problem on every convex set.

Key words: Equilibrium problem, Proper quasimonotone bifunctions, KKM maps, Generalized monotonicity

Introduction

By an equilibrium problem we understand the problem of finding

$$\overline{x} \in K \quad \text{such that} \quad f(\overline{x}, y) \geqslant 0, \quad \forall y \in K,$$

where K is a given set and $f : K \times K \to \mathbf{R}$ is a given function. This problem contains as special cases optimization problems, problems of Nash equilibria, complementarity problems, fixed point problems, and variational inequality; it unifies these problems in a convenient way, and many of the results obtained for one of these problems can be extended, with suitable modifications, to general equilibrium problems, thus obtaining wider applicability. Recently, many authors pointed out that also problems of practical interest in optimization, economics and engineering can be described by suitable equilibrium problems, and this explains the vast and increasing attention devoted to this subject.

This paper adapts some recent results from variational inequality problems to equilibrium problems. This is a very general phenomenon in this field. More specifically, we deal with necessary and sufficient conditions for the existence of solutions to equilibrium problems. In particular we consider the following problem find $\overline{x} \in K$:

$$F(\overline{x}, y) \geqslant \inf_{\triangle} F \quad \forall y \in K, \tag{EP}$$

and the closely related “dual”
find $\overline{y} \in K$:

$$F(x, \overline{y}) \leqslant \sup_{\triangle} F \quad \forall x \in K, \tag{DEP}$$

where, now and in the sequel, X denotes a topological Hausdorff vector space, K is a convex compact subset of X, $F : K \times K \to \mathbf{R}$, and $\triangle = \{(t, t), t \in K\}$. The compactness of K can be removed by assuming a suitable coercivity condition (see, for instance, [2, 14]).

If $F|_{\triangle} = 0$, these problems are well known and have been intensively studied by many authors (see [1–3, 14]). In particular, if $F(x, y) = \phi(y) - \phi(x)$, EP is a restatment of the minimization problem

$$\min \phi(x), \quad x \in K,$$

and if $F(x, y) = \langle A(x), y - x\rangle$, where $A : K \to X^*$ and $\langle\cdot,\cdot\rangle$ is the duality pair between X and X^*, EP is the well-known variational inequality problem (see [8]), and DEP is the more recently studied Minty variational inequality (see [7, 9, 10]).

Let us recall the definition of gap functions for EP and DEP, respectively:

$$m(x) = \inf_y F(x, y), \qquad s(y) = \sup_x F(x, y).$$

Since the inequality $m(x) \leqslant s(y)$ holds for every $x, y \in K$, a straightforward computation shows that for any function F it's true that

$$\sup_x \inf_y F(x, y) \leqslant \inf_y \sup_x F(x, y).$$

Moreover,

$$m(x) = \inf_y F(x, y) \leqslant F(x, x) \quad \forall x \in K,$$

and

$$s(y) = \sup_x F(x, y) \geqslant F(y, y) \quad \forall y \in K.$$

In particular, if $F|_{\triangle} = 0$, we obtain that

$$m(x) \leqslant 0 \leqslant s(y).$$

If we denote by $\overline{x}$ and $\overline{y}$ two solutions of EP and DEP, taking into account the definitions of m and s, it's easy to prove that the following chain of inequalities is satisfied

$$\inf_{\triangle} F \leqslant m(\overline{x}) \leqslant \sup_x m(x) \leqslant \inf_y s(y) \leqslant s(\overline{y}) \leqslant \sup_{\triangle} F.$$

We note that every points $\overline{x}$ and $\overline{y}$ satisfying the first and the last inequalities above are obviously solutions of EP and DEP, respectively. Notice that in the special case $F|_{\triangle} = 0$, since $\inf_{\triangle} F = \sup_{\triangle} F (= 0)$, all the inequalities above are equalities; in particular, if $\overline{x}$ and $\overline{y}$ are solutions of EP and DEP,

i) $m(\overline{x}) = 0 = s(\overline{y})$;

ii) $(\overline{x}, \overline{y})$ is a saddlepoint for the function F, since

$$F(x, \overline{y}) \leqslant F(\overline{x}, \overline{y})(= 0) \leqslant F(\overline{x}, y) \qquad \forall x, y \in K;$$

iii) the following minimax result holds

$$\sup_x \inf_y F(x, y) = \inf_y \sup_x F(x, y).$$

It's interesting to remark that if $F|_{\triangle} \neq 0$, then, in general, some of the inequalities above are strict.

EXAMPLE: Take the function $F : [0, 1] \times [0, 1] \to \mathbf{R}$ defined as follows:

$$F(x, y) = (x + y + 1)^{-1} a^{-|x-y|},$$

where $1 < a < \sqrt[3]{e}$. Trivial computations show that

$$\inf_{\triangle} F = \frac{1}{3}, \qquad \sup_{\triangle} F = 1,$$

and

$$m(x) = \frac{1}{x+2} a^{x-1}, \qquad s(y) = \frac{1}{y+1} a^{-y};$$

moreover, since $m(x) > 1/3$ and $s(y) < 1$ for all $x, y \in (0, 1)$, any $x \in (0, 1)$ is a solution of EP and any $y \in (0, 1)$ is a solution of DEP. In particular, for every $x, y \in (0, 1)$,

$$(\inf_{\triangle} F =)\frac{1}{3} < m(x) < \sup_x m(x) = \frac{1}{2a} = \inf s(y) < s(y) < \sup_{\triangle} F(= 1).$$

Let us notice that, in the case $F|_{\triangle} \neq 0$, the existence of solutions of both EP and DEP does not imply that a minimax result holds for the function F.

EXAMPLE: Take the function $F : [0, 1] \times [0, 1] \to \mathbf{R}$ defined as follows

$$F(x, y) = \begin{cases} 1 & x + y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Trivial computations show that, for every $x, y \in [0, 1]$,

$$\inf_{\triangle} F = 0, \quad \sup_{\triangle} F = 1, \quad m(x) = 0, \quad s(y) = 1,$$

therefore, any x solves EP and any y solves DEP, but we have that

$$\inf_x \sup_y F(x, y) > \sup_y \inf_x F(x, y).$$

In the case $F|_\triangle = 0$, the conditions $m(\overline{x}) = 0$ and $s(\overline{y}) = 0$ provide a complete characterization of the solutions of EP and DEP (see [10]). In the general case, if $m(\overline{x}) = \inf_\triangle F$ for some $\overline{x} \in K$, then $\overline{x}$ solves EP; if $s(\overline{y}) = \sup_\triangle F$ for some $\overline{y} \in K$, then $\overline{y}$ solves DEP. The first example shows that this is no longer a necessary condition, since there are solutions x and y of EP and DEP such that $m(x) > \inf_\triangle F$ and $s(y) < \sup_\triangle F$.

In this paper, we investigate a property of the bifunction F, called properly quasimonotonicity, strictly related to existence results for DEP.

In Section 1, we state some relationships between properly quasimonotone bifunctions and generalized monotone bifunctions.

In Section 2, the main result proves equivalent conditions, involving the properly quasimonotonicity, for the solvability of dual equilibrium problem on every closed, convex subset of K. We also improve a result on EP in [1].

For the definitions of generalized monotone bifunctions see, for instance, [1].

1. Properly Quasimonotone Bifunctions

The definition of properly quasimonotonicity for a bifunction turns out to be very useful when dealing with equilibrium problems.

Let $F : K \times K \to \mathbf{R}$;

DEFINITION 1.1. The function F is said to be *properly quasimonotone* (*pqm*) on $K \times K$ if for every finite set A of K, and for every $y \in \text{co}(A)$ the following inequality is satisfied

$$\min_{x \in A} F(x, y) \leqslant 0. \tag{1.1}$$

REMARK 1.1. If F is pqm, then $F|_\triangle \leqslant 0$; indeed, take $A = \{x\}$, for every $x \in K$; then $y = x$ and $F(x, x) \leqslant 0$.

The definition of pqm for bifunction is not new; it appears in [17], under the name of 0-diagonally quasiconcavity, and it is used in [16] to prove a result for the existence of solutions of a dual equilibrium problem. The name of proper quasimonotonicity is due to Daniilidis and Hadjisavvas; in [4], they called pqm an operator $T : X \to 2^{X^*}$ satisfying, for every $A = \{x_1, \dots, x_n\}$, for every $y \in \text{co}(A)$,

$$\forall x_i^* \in T(x_i): \quad (x_i^*, y - x_i) \leqslant 0, \quad \text{for some } i.$$

Let us recall that a bifunction F is said to be *pseudomonotone* on $K \times K$ if $F(x, y) \geqslant 0$ implies $F(y, x) \leqslant 0$, for every $x, y \in K$. The following conditions, easy to check, provide simple sufficient criteria for the proper quasimonotonicity of a bifunction:

PROPOSITION 1.1. *If F satisfies the assumptions*
i) $F(\cdot, y)$ *quasiconcave and* $F|_\triangle \leqslant 0$, *or*
ii) $F(x, \cdot)$ *quasiconvex (qcx) and F pseudomonotone, with* $F|_\triangle = 0$,
then F is pqm.

It's interesting to investigate the relationships existing between the concept of pqm, and those of generalized monotonicity, quasimonotonicity in particular. In the case of an operator $T : X \to 2^{X^*}$, is was shown in [4] that if T is pqm, then it is qm. Let us notice that, without specific assumptions on F, there is not a relationship between pqm and qm, as the following examples put in evidence:

EXAMPLES: The function $F : [0, 1] \times [0, 1] \to \mathbf{R}$ defined as follows

$$F(x, y) = \begin{cases} 1 & \text{if } y = 0, x > 0, \text{ or } x = 0, y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

is pqm, but is not qm; the function $F : [0, 1] \times [0, 1] \to \mathbf{R}$ defined as follows

$$F(x, y) = \begin{cases} 1 & \text{if } x + y < 1, y > x, \text{ or } x + y > 1, y < x \\ 0 & \text{otherwise,} \end{cases}$$

is qm, but is not pqm.

To state some relationships between the two concepts, we need some regularity assumptions. We recall that a function $f : K \to \mathbf{R}$ (K convex subset of X) is said to be *semistrictly quasiconcave (s.s.qcv) on K if for every $x_1, x_2 \in K$ such that $f(x_2) > f(x_1)$, the inequality $f((1 - \lambda)x_1 + \lambda x_2) > f(x_1)$ holds for every* $\lambda \in (0, 1)$.

A first result is the following

PROPOSITION 1.2. *Assume that $F(x, \cdot)$ is radially lower semicontinuous (r. lsc) and s.s. qcv on K, and F is pqm, with $F|_\triangle = 0$. Then, F is qm.*

Proof. Let $x, y \in K$ such that $F(x, y) > 0$, and assume by absurd that $F(y, x) > 0$. Denote by y_t the point $(1 - t)y + tx$. By the r. lsc we have that $(0 <)F(x, y) \leqslant \liminf_{t\to 0^+} F(x, y_t)$; in particular, for some $t \in (0, 1)$, $F(x, y_t) > 0$. From the pqm, then, $F(y, y_t) \leqslant 0$, and from the s.s. qcv il follows that $(0 \geqslant)F(y, y_t) > \min\{F(y, x), F(y, y)\} = 0$, a contradiction.

Another result, useful when $F(x, y) = h(x, y - x)$ and h is a generalized derivative, is provided by the following

PROPOSITION 1.3. *If F is pqm, and is 'positively homogeneous', then F is qm (positively homogeneous means that there exists $p > 0$ such that*

$$\lambda^p F(x, y) = F(x, x + \lambda(y - x))$$

for every $\lambda > 0$).

Proof. Indeed, let $F(x, y) > 0$, and consider any point $y_t = (1-t)x + ty$, $t \in (0, 1)$. By the positive homogeneity of F, since $F(x, y_t) = t^p F(x, y)$, then $F(x, y_t) > 0$. By the assumption of pqm, since $y_t \in \mathrm{co}\{x, y\}$, we obtain that $F(y, y_t) \leqslant 0$, that is $F(y, x) \leqslant 0$, thereby proving the thesis.

REMARK 1.2. In Propositions 1.2 and 1.3 the pqm property can be weakened by requiring that F is pqm 'along lines', that is for every $x_1, x_2 \in K$, and for every $\overline{y}$ in the segment $[x_1, x_2]$, the condition $F(x_1, \overline{y}) > 0$ implies $F(x_2, \overline{y}) \leqslant 0$. This condition is weaker than the pqm condition, as is proved by the following:

EXAMPLE: Let $F : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ be the function defined as follows

$$F(\underline{x}, \underline{y}) = \begin{cases} 1 - y_1 - y_2 & \text{if } \underline{x} = (0, 1) \\ y_1 & \text{if } \underline{x} = (0, 0) \\ y_2 & \text{if } \underline{x} = (1, 0) \\ 0 & \text{otherwise.} \end{cases}$$

This function is not pqm (indeed, taking $\underline{x}_1 = (0, 1)$, $\underline{x}_2 = (0, 0)$, $\underline{x}_3 = (1, 0)$, for any $\underline{y}$ in the relative interior of $\mathrm{co}(\underline{x}_1, \underline{x}_2, \underline{x}_3)$, we have that $F(\underline{x}_i, \underline{y}) > 0$), but it can be easily checked that F is pqm along lines.

It is proved in [4] that if $\phi : K \to \mathbf{R} \cup \{+\infty\}$ is lsc on K, there is equivalence between qm and pqm of $\partial\phi$, where ∂ denotes the Clarke–Rockafellar subdifferential. We can prove similar results for generalized derivatives of a radially lsc function. In particular, denote by $h(x, d)$ the lower (upper) Dini derivative of ϕ at x along the direction d. Let us recall (see [11]) that if ϕ is radially lsc, then ϕ is quasiconvex if and only if the lower (upper) Dini derivative $D_+\phi(x, y - x)$ ($D^+\phi(x, y - x)$) is qm as a function of (x, y). Moreover, if ϕ is quasiconvex, then

$$\begin{aligned} \phi(y) \leqslant \phi(x) &\Longrightarrow D_+\phi(x, y - x) \leqslant 0, \\ \phi(y) \leqslant \phi(x) &\Longrightarrow D^+\phi(x, y - x) \leqslant 0. \end{aligned} \tag{1.2}$$

We have the following:

PROPOSITION 1.4. *Suppose that ϕ is radially lower semicontinuous on the convex set K. Then $h(x, y-x) = D^+\phi(x, y-x)$ (or $D_+\phi(x, y-x)$) is quasimonotone if and only if it is properly quasimonotone.*

Proof. The 'if' part follows from Proposition 1.3. Let us show the 'only if' part. By absurd, suppose that there exists $x_1, \ldots, x_n$ and $\tilde{y} \in \mathrm{co}(x_1, \ldots, x_n)$ such that

$$h(x_i, \tilde{y} - x_i) > 0, \quad \forall i = 1, 2, \ldots, n.$$

From (1.2), $f(x_i) < f(\tilde{y})$. In particular, none of the points x_i can be a maximum point for f on the convex set $\mathrm{co}(x_1, \ldots, x_n)$. But this contradicts the well known property of vertex–maximum of quasiconvex function over a polyedron (see, for instance, [13]). Therefore, h is pqm.

Following the definitions of generalized convexity with respect to a bifunction given in [11], Proposition 1.4 can be extended in a natural way to more general settings.

2. Equilibrium problems for pqm bifunctions

Let us go back to the dual equilibrium problem in order to motivate the interest in the pqm property of a bifunction. Given the bifunction $F : K \times K \to \mathbf{R}$, denote by $G : K \to 2^K$ the set-valued map defined by

$$G(x) = \{y \in K : F(x, y) \leqslant \sup_{\triangle} F\}.$$

Since the set $\cap_{x\in K} G(x)$ is the set of the solutions of DEP, any results about the existence of solutions of DEP can be restated as a result of nonemptiness of the intersection above. A useful tool to investigate existence results of equilibrium problems is the Ky Fan lemma, involving the KKM property of multivalued maps.

DEFINITION 2.1. Let D be a nonempty subset of X. A function $\Phi : D \to 2^D$ is called KKM if for any $x_1, x_2, \ldots, x_n \in D$ and for any $y \in \mathrm{co}(x_1, x_2, \ldots, x_n)$ one has $y \in \cup_i \Phi(x_i)$.

LEMMA 2.1 (Ky Fan). *Let D be a nonempty subset of X, and $\Phi : D \to 2^D$ be a KKM function. Assume that $\Phi(y)$ is closed for each $y \in D$, and $\Phi(y_0)$ is compact for some $y_0 \in D$. Then $\cap_{y\in D}\Phi(y) \neq \emptyset$.*

It turns out that the pqm property of F is the translation of the KKM property of the map G previously defined. This explains the interest in the definition.

The following proposition provides a sufficient condition to solve DEP.

PROPOSITION 2.1. *Let $F : K \times K \to \mathbf{R}$ be a function lsc in the second variable, and such that $F - \sup_{\triangle} F$ is pqm. Then $\cap_{x\in K} G(x) \neq \emptyset$. If F is quasiconvex in the second variable, then the set of solutions is convex, and if $\sup_{\triangle} F - F$ is strictly pseudomonotone, then $\cap_{x\in K} G(x)$ is a singleton.*

Proof. The first part of the proof follows as a special case from th. 2.2 in [16]. To show uniqueness of the solution, let $\overline{x}, \hat{x} \in \cap_{x\in K} G(x)$, $\overline{x} \neq \hat{x}$. Both of them are solutions of DEP, therefore

$$F(\overline{x}, \hat{x}) \leqslant \sup_{\triangle} F, \qquad F(\hat{x}, \overline{x}) \leqslant \sup_{\triangle} F. \tag{2.3}$$

But, from the strict pseudomonotonicity,

$$F(\overline{x}, \hat{x}) \leqslant \sup_{\triangle} F \Longrightarrow F(\hat{x}, \overline{x}) > \sup_{\triangle} F,$$

contradicting (2.1). Finally, the convexity of the solution set given by $\cap_{x\in K} G(x)$ follows immediately from the convexity of each of $G(x)$.

To state the result proving that the pqm property cannot be weakened in order to solve DEP, we need the following topological lemma:

LEMMA 2.2 ([15], Corollary 6.5.1). *Let C be a convex subset of $\mathbf{R}^n$, and let M be an affine set wich contains a point of ri C. Then*

$$ri\ (M \cap C) = M \cap ri\ C, \qquad cl\ (M \cap C) = M \cap cl\ C.$$

(ri C denotes the relative interior of C and cl C the closure of C.)

The following theorem essentially states that the condition of proper quasi-monotonicity is quite sharp in solving dual equilibrium problems. Before stating this main result, we wish to recall that an analogue result was proved in the different framework of the Minty variational inequality for multivalued maps in [9]. It might be interesting to draw the attention to that both John's and the present result are of Martos type, in the following sense: ϕ is quasiconvex on a convex set C if and only if it takes a vertex–maximum over any compact polyedral subset of C; indeed, if we put $F(x, y) = \phi(y) - \phi(x)$, then F is pqm if and only if ϕ is quasiconvex.

THEOREM 2.1 *Let $F : K \times K \to \mathbf{R}$ be a function lsc and qcx in the second variable; if $G(x)$ denotes the set $\{y : F(x, y) \leqslant \sup_\triangle F\}$, then the following conditions are equivalent:*

i) $F - \sup_\triangle F$ is pqm;

ii) for any finite set $A \subseteq K$ there exists $y \in \mathrm{co}(A)$ such that $y \in \cap_{x\in A} G(x)$;

iii) DEP has a solution on every compact convex subset of K.

Proof. We prove the equivalence between i), ii) and iii) by showing that

i) $\Longrightarrow$ ii) $\Longrightarrow$ iii) $\Longrightarrow$ i).

i) $\Longrightarrow$ ii): let A be a finite subset of K, and denote by C the closed convex set co(A). Consider the function F defined on $C \times C$; since, by i), $F - \sup_\triangle F$ is pqm on $C \times C$, and lsc in the second component, then, by Proposition 2.1, $\cap_{x\in C}(G(x) \cap C) \neq \emptyset$. In particular, there exists $\overline{x} \in C$ such that $\overline{x} \in \cap_{x\in A} G(x)$, thereby proving ii);

ii) $\Longrightarrow$ iii): consider a convex compact subset C of K. We shall prove that

$$\cap_{x\in C}\ (G(x) \cap C) \neq \emptyset.$$

Take $A = \{x_1, x_2, \ldots, x_n\} \subseteq C$; from ii), there exists $\overline{x}$ $\in$co$(A) \subseteq C$ such that $\overline{x} \in \cap_1^n G(x_i)$. This implies that for any $\{x_1, x_2, \ldots, x_n\} \subseteq C$ the set $\cap_1^n (G(x_i) \cap C)$ is nonempty. Since $G(x) \cap C$ is compact for every $x \in C$, then, by a well known result, $\cap_{x\in C}(G(x) \cap C) \neq \emptyset$, thereby proving iii);

iii) $\Longrightarrow$ i): we shall prove this implication by induction on n. Indeed, since $F(x, x) \leqslant \sup_\triangle F$ for every $x \in K$, i) holds for $n = 1$. Let us assume that i) holds for any $n - 1$ points in K, and consider the finite set $A = \{x_1, x_2, \ldots, x_n\}$. Take $\overline{y} \in C$ =co(A). If $\overline{y} = \sum_1^m \lambda_{i_j} x_{i_j}$ with $m < n$, then, by the hypothesis of induction, since $\overline{y}$ belongs to a simplex of dimension less than n, there exists i_j such that $F(x_{i_j}, y) \leqslant \sup_\triangle F$. Otherwise, suppose that $\lambda_i \neq 0$ for every i; in this case, $\overline{y}$ is an interior point of the simplex generated by $\{x_1, x_2, \ldots, x_n\}$. Denote by $\overline{z}$ a solution of DEP on the set C, and assume that $\overline{z} \neq \overline{y}$ (if $\overline{z} = \overline{y}$, then i) trivially holds). Denote by aff $(\overline{z}, \overline{y})$ the affine space (line) containing $\overline{z}$ and $\overline{y}$. Since $\overline{y} \in$ ri C, then Lemma 2.2 can be applied to the sets C and $M =$ aff $(\overline{z}, \overline{y})$ considered as subsets of the finite–dimensional space generated by $\{x_1, x_2, \ldots, x_n\}$. It follows that there exists $\overline{w} \in$ aff $(\overline{z}, \overline{y}) \cap \partial C$ such that $\overline{w} \neq \overline{z}$, $\overline{w} = \sum_1^m \lambda_{i_j} x_{i_j}$ $(m < n)$ and $\overline{y} \in [\overline{z}, \overline{w}]$. Since $\overline{w} \in \mathrm{co}(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$, by the hypothesis of induction there exists i_k such that $F(x_{i_k}, \overline{w}) \leqslant \sup_\triangle F$. Moreover, $F(x_i, \overline{z}) \leqslant \sup_\triangle F$ $(i = 1, 2, \ldots, n)$. Therefore, by the convexity of the set $G(x_{i_k})$, since $\overline{w}, \overline{z} \in G(x_{i_k})$, the whole segment $[\overline{z}, \overline{w}]$ belongs to this set; in particular, $\overline{y} \in G(x_{i_k})$, thereby showing that i) holds for a set of n points.

Under suitable conditions, the existence of solutions of DEP implies the solvability of EP. We need the following

DEFINITION 2.2. A real function f defined on a convex subset K of X is said to be *hemicontinuous* if $\lim_{t\to 0^+} f(tx + (1-t)y) = f(y)$, for each $x, y \in K$.

PROPOSITION 2.2. *Let $F : K \times K \to \mathbf{R}$ be a pqm function with $F|_\triangle = 0$, semistrictly (ss) qcx and lsc in the second variable, and hemicontinuous in the first one. Then EP has a solution.*

Proof. From Proposition 2.1, there exists $\overline{x}$ solution of DEP. We prove that $\overline{x}$ is a solution of EP. Take any $x \in K$, and define $x_t = (1-t)\overline{x} + tx$ for $t \in [0, 1]$. Since F is qcx in the second component,

$$0 = F(x_t, x_t) \leqslant \max\{F(x_t, \overline{x}), F(x_t, x)\}.$$

If $F(x_t, x) < F(x_t, \overline{x})$, then

$$F(x_t, \overline{x}) \leqslant F(x_t, x_t) \leqslant F(x_t, \overline{x}),$$

so that $F(x_t, x_t) = F(x_t, \overline{x})$, contradicting the ss qcx. Then

$$F(x_t, \overline{x}) \leqslant F(x_t, x_t) \leqslant F(x_t, x), \qquad \forall t \in [0, 1].$$

Taking $t \to 0^+$, by the hemicontinuity we have $F(\overline{x}, x) \geqslant F(\overline{x}, \overline{x}) = 0$, thereby $\overline{x}$ solves EP.

To conclude, we show that the assumptions of Proposition 2.2 are weaker than those in [1]. To this aim, let us consider the following:

COUNTEREXAMPLE: Consider $K = [0, 1]$ and the function $F : K \times K \to \mathbf{R}$ defined as follows

$$F(x, y) = \begin{cases} 2x - 2y - 1 & \text{if } 1/2 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x - 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

This function is trivially not pseudomonotone, but it is pqm, quasiconvex and continuous on $[0, 1] \times [0, 1]$.

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References

1. Bianchi, M. and Schaible, S. (1996), Generalized Monotone Bifunctions and Equilibrium Problems, *J. Optim. Theory Appl.* 90: 31–43.
2. Blum, E. and Oettli, W. (1994), From Optimization and Variational Inequalities to Equilibrium Problems, *The Mathematics Student* 63: 123–145.
3. Chadli, O., Chbani, Z. and Rihai, H. (2000) Equilibrium Problems with Generalized Monotone Bifunctions and Applications to Variational Inequalities, *J. Optim. Theory Appl.* 105: 299–323.
4. Daniilidis, A. and Hadjisavvas, N. (2000), On Generalized Cyclically Monotone Operators and Proper Quasimonotonicity, *Optimization* 47: 123–135.
5. Dugundji, J. and Granas, A. (1978), KKM Maps and Variational Inequalities, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 5(4): 679–682.
6. Fan, K. (1961), A Generalization of Tychonoff's Fixed point Theorem, *Math. Ann.* 142: 305–310
7. Giannessi, F. (1997), On Minty Variational Principle, in: F. Giannessi, S. Komlosi and T. Rapsack (eds), *New Trend in Mathematical Programming*, Kluwer Academic Publishers, Dordrecht, The Netherlands.
8. Harker, P.T. and Pang, J.S. (1990), Finite Dimensional Variational Inequalities and Nonlinear Complementarity Problems: a Survey of Theory. Algorithms and Applications, *Math. Programming* 48: 161–220.
9. John, R. (1999), A Note on Minty Variational Inequalities and Generalized Monotonicity (preprint).
10. Komlosi, S. (1999), On the Stampacchia and Minty Variational Inequalities, in: G. Giorgi and F. Rossi (eds), *Generalized Convexity and Optimization for Economic and Financial Decisions*, Pitagora Editrice.
11. Komlosi, S. (1995), Generalized Monotonicity and Generalized Convexity, *J. Optim. Theory Appl.* 84: 361–376.
12. Komlosi, S. (1994), Generalized Monotonicity in Non-Smooth Analysis, in: S. Komlosi, T. Rapcsak and S. Schaible (eds), *Generalized Convexity*, Springer-Verlag.
13. Martos, B. (1975), *Nonlinear Programming Theory and Methods*, North Holland, Amsterdam.
14. Mosco, U. (1976), Implicit Variational Problems and Quasivariational Inequalities, *Lectures Notes in Math.* 543: 83–156 Springer-Verlag.
15. Rockafellar, R.T. (1972), *Convex Analysis*, Princeton University Press, Princeton, New Jersey.
16. Tan, K.K. and Yuan, Z.Z. (1993), A Minimax Inequality with Applications to Existence of Equilibrium Points, *Bull. Austral. Math. Soc.* 47: 483–503.
17. Zhou, J.X. and Chen, G. (1998), Diagonal Convexity Conditions for Problems in Convex Analysis and Quasi-Variational Inequalities, *J. Math. Anal. Appl.* 132: 213–225.